Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 1

30 October 2023

Zone A afternoon | **Zone B** afternoon | **Zone C** afternoon

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all questions.
- Answers must be written within the answer boxes provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

8823–7201

Answers must be written within the answer boxes provided. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 7]

The growth of a particular type of seashell is being studied by Manon. At the end of each month Manon records the increase in the width of a seashell since the end of the previous month.

She models the monthly increase in the width of the seashell by a geometric sequence with common ratio 0.8. In the first month, the width of the seashell increases by $4\,\text{mm}$.

(a) Find by how much the width of the seashell will increase during the third month, according to her model. [2]

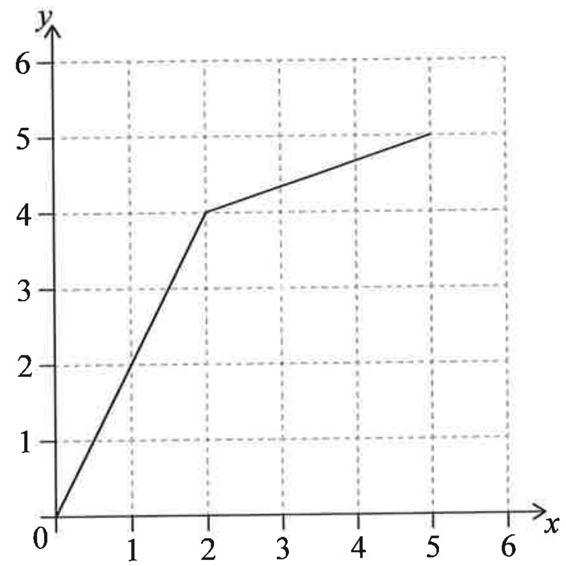
(b) Find the total increase in the width of the seashell, predicted by Manon's model, during the first year. [2]

Manon's seashell had a width of $25\,\text{mm}$ at the beginning of the first month.

(c) Find the maximum possible width of the seashell, predicted by Manon's model. [3]

2. [Maximum mark: 7]

The graph of the function f is given in the following diagram.



(a) Write down $f(2)$. [1]

(b) On the axes, sketch $y = f^{-1}(x)$. [2]

The function g is defined as $g(x) = 3x - 1$.

(c) Find an expression for $g^{-1}(x)$. [2]

(d) Find a value of x where $f^{-1}(x) = g^{-1}(x)$. [2]

3. [Maximum mark: 6]

The Great Pyramid of Giza is the oldest of the Seven Wonders of the Ancient World. When it was built, 4500 years ago, the measurements of the pyramid were in Royal Egyptian Cubits (REC).

Viktor reads online that $1\,\text{REC}$ is equal to 0.52 metres, rounded to two decimal places.

(a) Write down the upper and lower bounds of $1\,\text{REC}$ in metres. [2]

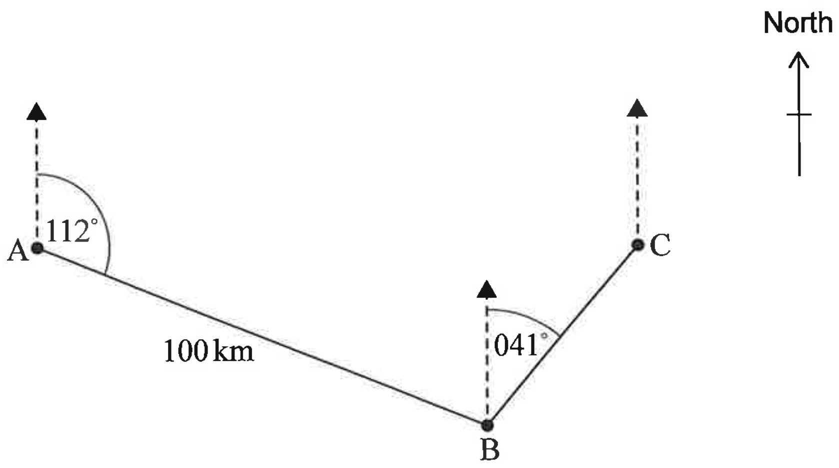
The Great Pyramid of Giza has a square base with side lengths of $440\,\text{REC}$ and a height of $280\,\text{REC}$. Viktor assumes that these two measurements are exact and that the Great Pyramid can be modelled as a square-based pyramid with smooth faces.

(b) Find the minimum possible volume of the pyramid in cubic metres. [4]

4. [Maximum mark: 6]

Jason sails his boat from point A for a distance of $100\,\text{km}$, on a bearing of 112°, to arrive at point B. He then sails on a bearing of 041° to point C. Jason's journey is shown in the diagram.

diagram not to scale



(a) Find $\text{A}\hat{\text{B}}\text{C}$. [2]

Point C is directly east of point A.

(b) Calculate the distance that Jason sails to return directly from point C to point A. [4]

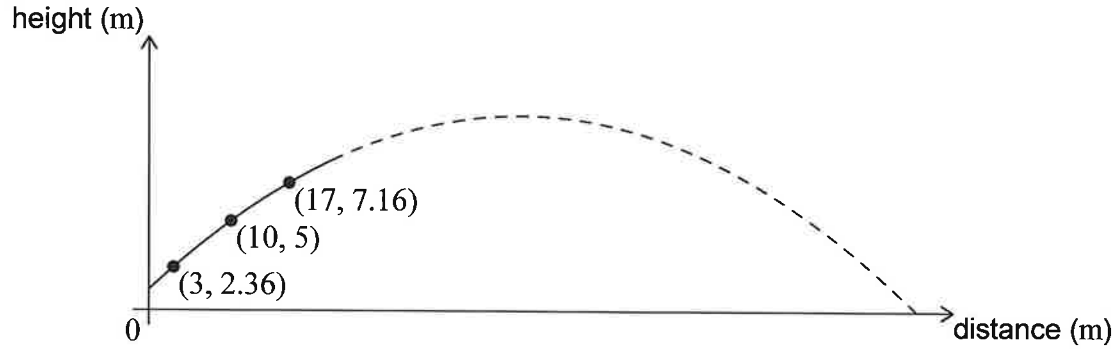
5. [Maximum mark: 9]

A sports player on a horizontal athletic field hits a ball. The height of the ball above the field, in metres, after it is hit can be modelled using a quadratic function of the form $f(x) = ax^2 + bx + c$, where x represents the horizontal distance, in metres, that the ball has travelled from the player.

A specialized camera tracks the initial path of the ball after it is hit by the player. The camera records that the ball travels through the three points $(3, 2.36)$, $(10, 5)$ and $(17, 7.16)$, as shown in **Diagram 1**.

diagram not to scale

Diagram 1



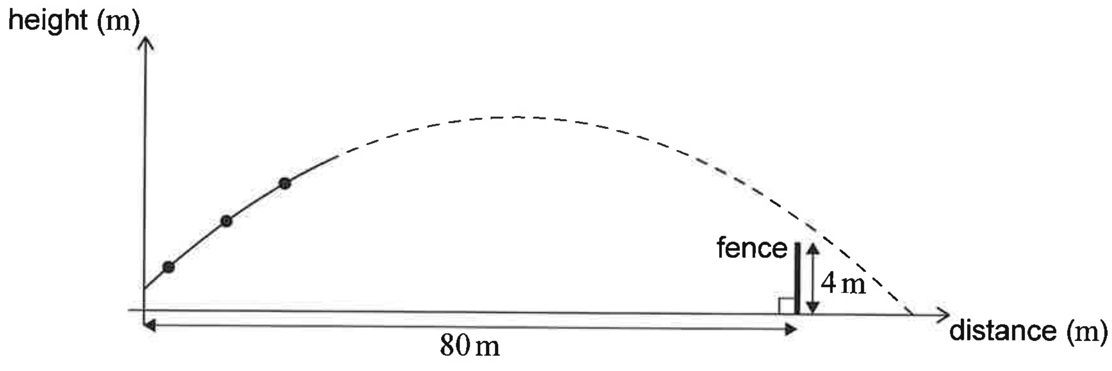
(a) Use the coordinates $(3, 2.36)$ to write down an equation in terms of a, b, and c. [1]

(b) Use your answer to part (a) and two similar equations to find the equation of the quadratic model for the height of the ball. [3]

A 4-metre-high fence is 80 metres from where the player hit the ball, as shown in **Diagram 2**.

diagram not to scale

Diagram 2



(c) Show that the model predicts that the ball will go over the fence. [3]

(d) Find the horizontal distance that the ball will travel, from the player until it first hits the ground, according to this model. [2]

(This question continues on the following page)

(Question 5 continued)

6. [Maximum mark: 8]

On the following Voronoi diagram, the coordinates of three farmhouses are $\mathrm{A}(0, 3)$, $\mathrm{B}(8, 3)$ and $\mathrm{C}(8, 13)$, where distances are measured in kilometres. Each farmhouse owns the land that is closest to it, and their boundaries are defined by the dotted lines on **Diagram 1**.

Diagram 1

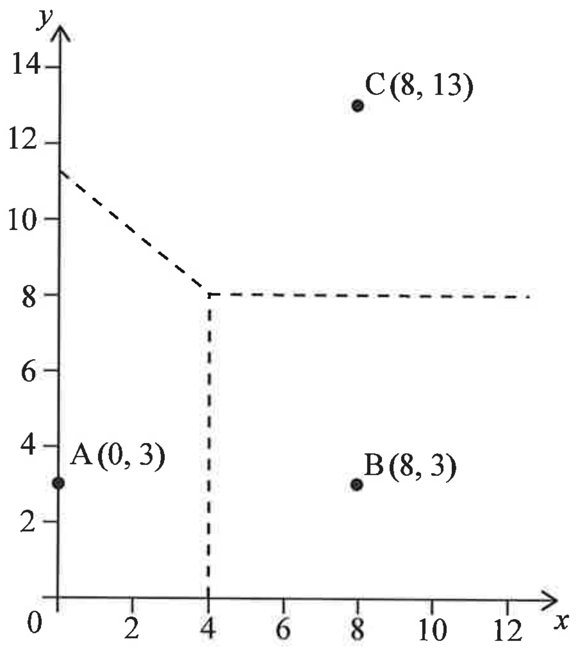
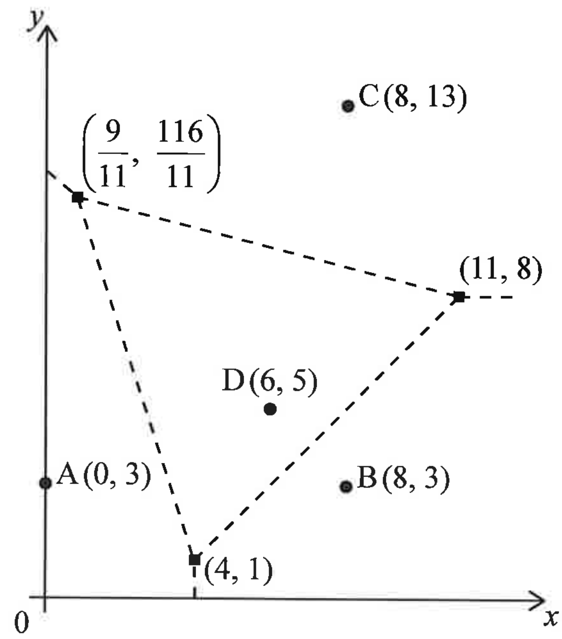


Diagram 2



To provide water to the farms it is decided to construct a well at the point where the boundaries meet on **Diagram 1**.

(a) Write down the coordinates of this point. [1]

(b) Find the equation of the perpendicular bisector of $[\mathrm{AC}]$. [3]

An additional farmhouse $\mathrm{D}(6, 5)$ is built on the land. The Voronoi diagram has been redrawn to show the new boundaries. The coordinates of the vertices of these boundaries are indicated on **Diagram 2**.

A wind turbine is to be built at one of the vertices.

(c) The wind turbine should be as far from the nearest farmhouses as possible.

(i) By calculating appropriate distances, find the location of the wind turbine.

(ii) Hence, write down the distance of the wind turbine to the nearest farmhouse. [4]

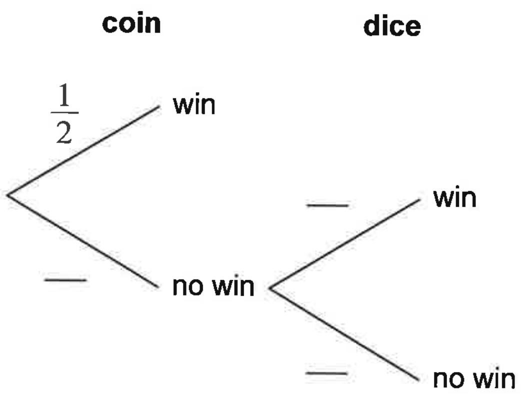
(This question continues on the following page)

(Question 6 continued)

7. [Maximum mark: 7]

Michèle is playing a game. In the game, she must first flip a fair coin which will result in the coin landing on heads or tails. If the coin lands on heads, then she wins a prize. If it lands on tails, then she has another chance but this time she must roll a fair six-sided dice and get a five or six in order to win a prize.

(a) Complete the tree diagram by writing in the three missing probabilities. [2]



(b) Find the probability that Michèle does **not** win a prize. [2]

(c) Given that Michèle won a prize, find the probability that the coin landed on heads. [3]

8. [Maximum mark: 6]

Given $z=\sqrt{3}-\mathrm{i}$.

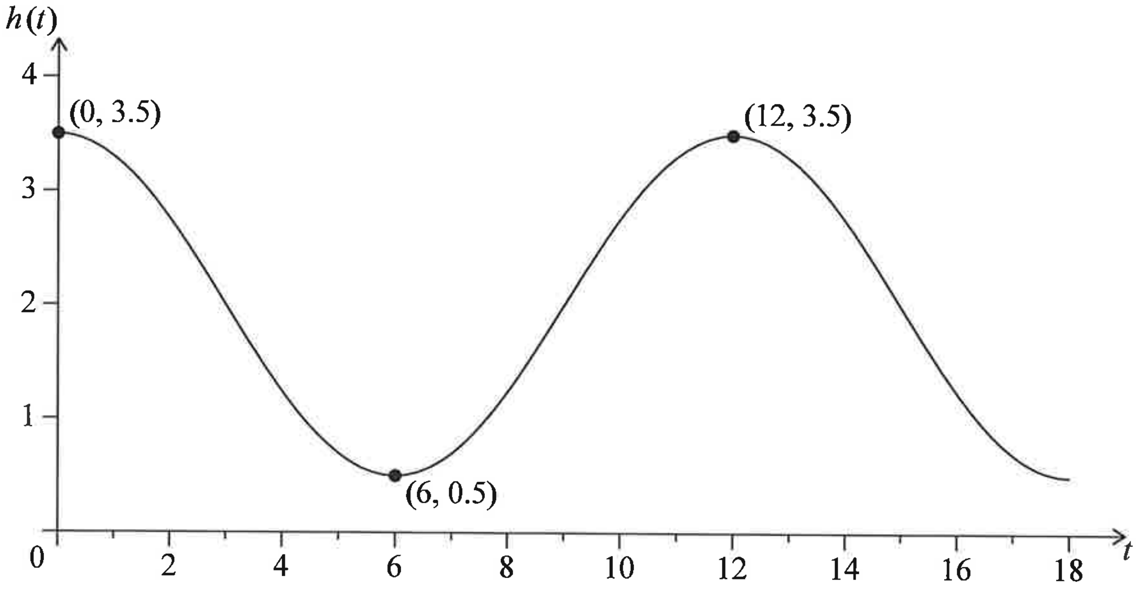
(a) Write z in the form $z=r\mathrm{e}^{\theta\mathrm{i}}$, where $r\in\mathbb{R}^+$, $-\pi<\theta\leq\pi$. [2]

Let $z_1=\mathrm{e}^{2t\mathrm{i}}$ and $z_2=2\mathrm{e}^{\left(2t-\frac{\pi}{6}\right)\mathrm{i}}$.

(b) Find $\mathrm{Im}(z_1+z_2)$ in the form $p\sin(2t+q)$, where $p>0$, $t\in\mathbb{R}$ and $-\pi\leq q\leq\pi$. [4]

9. [Maximum mark: 8]

Joon is a keen surfer and wants to model waves passing a particular point P, which is off the shore of his favourite beach. Joon sets up a model of the waves in terms of $h(t)$, the height of the water in metres, and t, the time in seconds from when he begins recording the height of the water at point P.



The function has the form $h(t) = p\cos\left(\frac{\pi}{6}t\right) + q$, $t \geq 0$.

(a) Find the values of p and q. [2]

(b) Find

(i) $h'(t)$.

(ii) $h''(t)$. [3]

Joon will begin to surf the wave when the rate of change of h with respect to t, at P, is at its maximum. This will first occur when $t = k$.

(c) (i) Find the value of k.

(ii) Find the height of the water at this time. [3]

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(Question 9 continued)

10. [Maximum mark: 7]

The decay of a chemical isotope over five years is recorded in **Table 1**. The mass of the chemical M is measured to the nearest gram at the beginning of each year t of the experiment.

Table 1

Time t (years)	1	2	3	4	5
Mass M (grams)	1000	660	517	435	381

It is believed that the decay of the isotope can be modelled by an equation of the form $M = a \times t^b$.

(a) Use power regression on your graphic display calculator to find the value of a and the value of b. [2]

The values of t and M can be transformed such that $x = \ln t$ and $y = \ln M$. **Table 2** shows data for x and y to three decimal places.

Table 2

x	0	0.693	1.099	1.386	1.609
y	6.908	6.492	6.248	6.075	5.943

(b) Find the linear regression equation of y on x, in the form $y = cx + d$. Give the values of c and d to three decimal places. [2]

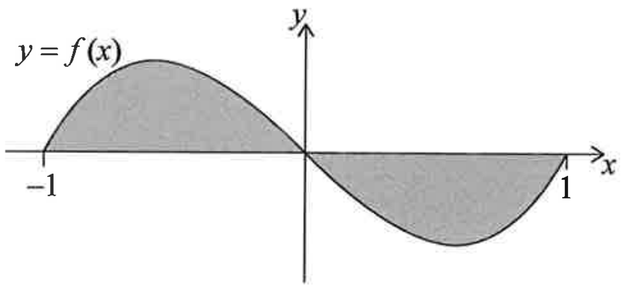
(c) Hence, show that this linear regression is equivalent to the power regression found in part (a). [3]

(This question continues on the following page)

(Question 10 continued)

11. [Maximum mark: 7]

Consider the function $f(x) = x^3 - x$, for $-1 \le x \le 1$. The shaded region, R, is bounded by the graph of $y = f(x)$ and the x-axis.

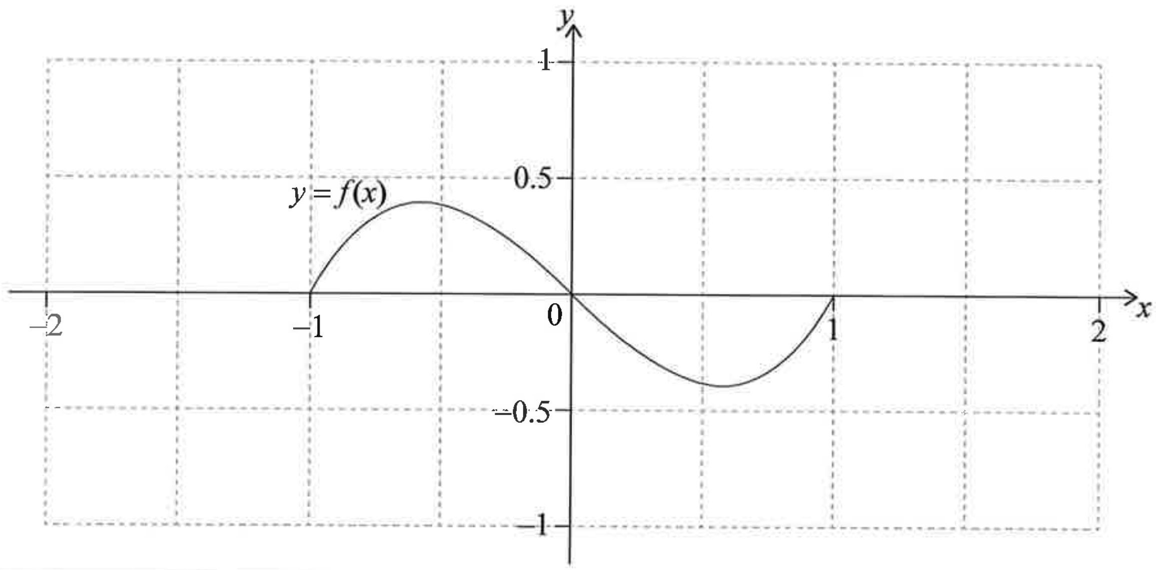


(a) (i) Write down an integral that represents the area of R.

(ii) Find the area of R. [2]

Another function, g, is defined such that $g(x) = 2f(x-1)$.

(b) On the following set of axes, the graph of $y = f(x)$ has been drawn. On the same set of axes, sketch the graph of $y = g(x)$. [2]



The region R from the original graph $y = f(x)$ is rotated through 2π radians about the x-axis to form a solid.

(c) Find the volume of the solid. [3]

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(Question 11 continued)

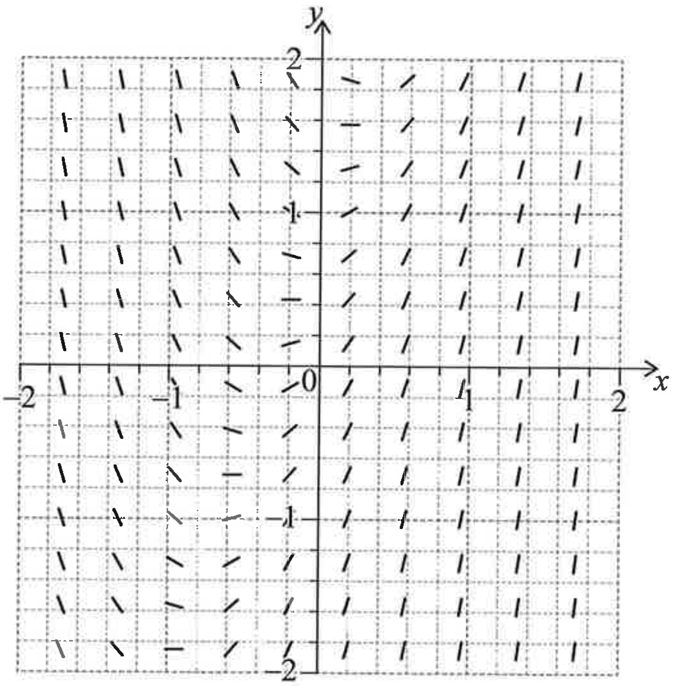
12. [Maximum mark: 4]

Consider the differential equation $\frac{dy}{dx} = 3x - y + 1$.

(a) Find the equation of the tangent to the solution curve at the point $(-1, -1)$ in the form $ax + by + c = 0$. [2]

The slope field for this differential equation is shown in the following diagram.

(b) Sketch the solution curve that passes through the point $(-1, -1)$. [2]



13. [Maximum mark: 8]

The velocity v of a particle at time t, as it moves along a straight line, can be modelled by the piecewise function

$$v(t) = \begin{cases} u_1(t), & 0 \le t \le T \\ u_2(t), & t \ge T \end{cases}$$

where $u_1(t) = 2t^2 - t^3$ and $u_2(t) = 8 - 4t$. It is required that $u_1(T) = u_2(T)$.

(a) Find the value of T. [2]

(b) Show that $u_1{}'(T) = u_2{}'(T)$. [2]

The displacement of the particle at time $t = 0$ is zero.

(c) Find the time when the particle returns to its initial position. [4]

14. [Maximum mark: 7]

A straight line L has vector equation $\boldsymbol{r} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and point Q has coordinates $(11, -1, 3)$.

Point P is the point on L closest to Q.

(a) Find the coordinates of P. [4]

(b) Find a vector that is perpendicular to both L and the line passing through points P and Q. [3]

15. [Maximum mark: 7]

The eating habits of students in a school are studied over a number of months. The focus of the study is whether non-vegetarians become vegetarians, and whether vegetarians remain vegetarians.

Each month, students choose between the vegetarian or non-vegetarian lunch options. Once they have chosen for the month, they cannot change the option until the next month.

In any month during the study, it is noticed that the probability of a non-vegetarian becoming vegetarian the following month is 0.1, and that the probability of a vegetarian remaining a vegetarian the following month is 0.8.

This situation can be represented by the transition matrix

$$\boldsymbol{T} = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix}.$$

(a) Interpret the value 0.9 in $\boldsymbol{T}$ in terms of the changes in the eating habits of the students in the school. [1]

(b) Find the eigenvalues of matrix $\boldsymbol{T}$. [3]

One of the eigenvectors of $\boldsymbol{T}$ is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(c) Find another, non-parallel, eigenvector and interpret it in context. [3]

16. [Maximum mark: 6]

When Jef plays basketball, the number of shots he takes during any 6 minutes of play can be modelled by a Poisson distribution with mean 2.5.

(a) Find the probability that Jef takes less than 7 shots during any 12 minutes of play. [2]

It can be assumed that the outcomes of the shots are independent of each other, and the probability of success of a shot is constant. The probability that Jef is successful with a shot is 0.4.

It can be assumed that the probability of Jef's success with a shot is independent of the number of shots that he takes.

(b) Find the probability that during any 6 minutes of play Jef takes fewer than 4 shots and is successful at least once. [4]

References:

3. syaber, n.d. *The Great Pyramid of Giza*. [image online] Available at: https://www.gettyimages.co.uk/detail/photo/the-great-pyramid-of-giza-royalty-free-image/1354416052 [Accessed 6 February 2023]. SOURCE ADAPTED.

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Answers written on this page
will not be marked.