Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 2

25 October 2024

Zone A morning | **Zone B** morning | **Zone C** morning

2 hours

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

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15 pages

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Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 14]

A survey was answered by 20 000 expatriates (people living in a country that is not their own). The data ranked countries in order of the country they felt was best for expatriates. The highest-ranked country was Switzerland.

These results were compared to happiness scores taken from *The World Happiness Report 2022*. The following table shows this data for the top 10 expatriate countries.

Country	Switzerland	New Zealand	Spain	Australia	Cyprus	Portugal	Ireland	United Arab Emirates	France	Netherlands
Expatriate country rank	1	2	3	4	5	6	7	8	9	10
Happiness score	7.5	7.2	6.5	7.2	6.2	6.0	7.0	6.6	6.7	7.4

(a) For the **happiness score**, find

(i) the upper quartile

(ii) the interquartile range. [4]

(b) Show that Switzerland's happiness score is not an outlier for this data. [3]

(This question continues on the following page)

(Question 1 continued)

The happiness scores were ranked to calculate Spearman's rank correlation coefficient, r_s. These ranks are shown in the following table.

Country	Switzerland	New Zealand	Spain	Australia	Cyprus	Portugal	Ireland	United Arab Emirates	France	Netherlands
Happiness score	7.5	7.2	6.5	7.2	6.2	6.0	7.0	6.6	6.7	7.4
Happiness rank	1	a	b	c	9	10	5	7	6	2

(c) Write down the value of

(i) a

(ii) b

(iii) c. [3]

(d) (i) Find r_s.

(ii) If France's happiness score is upgraded to 6.9, explain why the value of r_s does not change. [3]

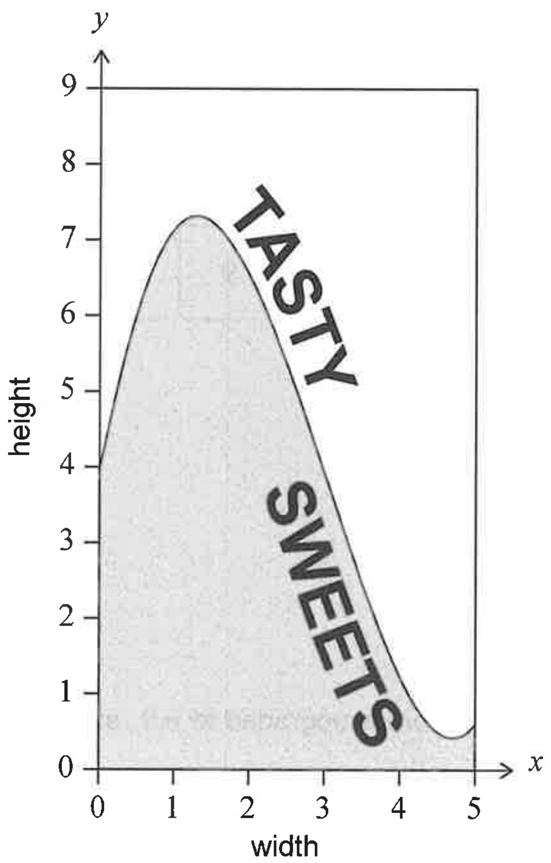
Jose concludes from this data that countries with high happiness scores are likely to be favourite expatriate countries.

(e) State, with a reason, whether Jose's conclusion is appropriate. [1]

2. [Maximum mark: 18]

Sweets are sold in cylindrical containers. A new label for the container is being considered. The label will be a rectangle that is $5\,\text{cm}$ wide and $9\,\text{cm}$ high.

The design on the label is a curve, as shown on the following axes, where one unit represents $1\,\text{cm}$ for both axes.



The values in the table approximate points on the curve, correct to one decimal place.

Width, x	0	1	2	3	4	5
Height, y	4	7.3	6.7	4.0	1.3	0.7

(a) Use the trapezoidal rule with five intervals, and the values given in the table, to estimate the shaded area below the curve. [3]

The curve used in the label design can be modelled by:

$$y = \frac{x^3}{3} - 3x^2 + 6x + 4\text{, for } 0 \le x \le 5.$$

(b) Use this equation to find the area of the shaded region. [2]

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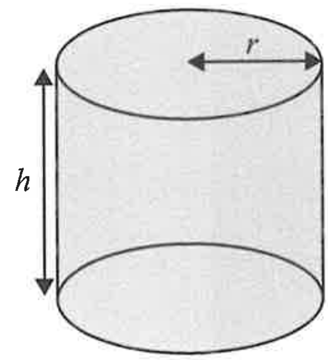
(Question 2 continued)

The sweets are sold in closed cylindrical containers, with radius r and height h.

diagram not to scale



The whole container is made from one type of material, and it is assumed that the thickness of the material is negligible.

Each container has a volume of $600\,\text{cm}^3$.

(c) Write down an equation, in terms of r and h, that shows this information. [1]

The amount of material used for each container can be modelled by the external surface area of the container.

The external surface area, A, of the container can be expressed as

$$A = 2\pi r^2 + \frac{k}{r}, \text{ where } r > 0.$$

(d) Find the value of k. [4]

(e) (i) Find $\frac{\mathrm{d}A}{\mathrm{d}r}$.

(ii) Given that A has a minimum value, find the value of r that will minimize the material used. [5]

The containers are made so that the surface area is minimized. The $5\,\text{cm}$ by $9\,\text{cm}$ rectangular label is to be glued to the curved surface of the container.

(f) Show that the label will fit on the container. [3]

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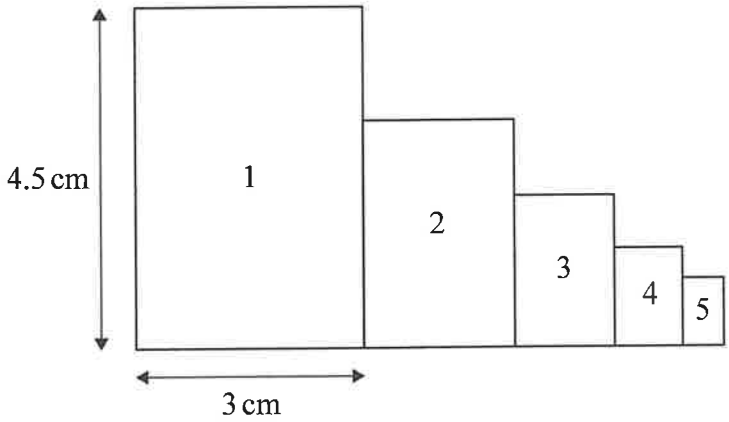
3. [Maximum mark: 15]

Ayaka is creating a design made from a sequence of rectangles. The diagram shows part of her design, using 5 rectangles.

diagram not to scale



The first rectangle has the following dimensions: height 4.5 cm and width 3 cm.

The dimensions of each subsequent rectangle are $\frac{2}{3}$ of the dimensions of the previous rectangle.

(a) Calculate the width of the 5th rectangle. [2]

(b) Calculate the total width of the design that uses 5 rectangles. [2]

Ayaka continues to add rectangles to her design.

(c) Find the smallest total width that her design cannot exceed. [3]

The width of Ayaka's final design must be at least 8.5 cm and use the least number of rectangles.

(d) Find the total number of rectangles in her final design. [3]

The decreasing **areas** of the rectangles form a geometric sequence.

(e) Find the common ratio for this sequence of areas. [2]

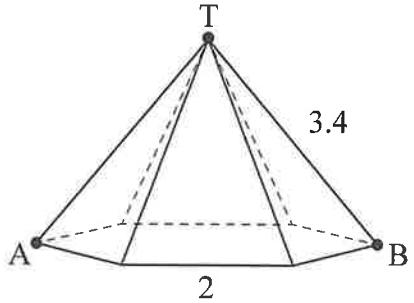
(f) Find the total area of Ayaka's final design. [3]

4. [Maximum mark: 17]

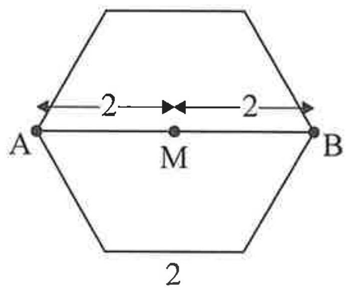
Gaurika is designing a tent in the shape of a right pyramid with a regular hexagonal base, centre M. The length of each side of the base is $2\,\mathrm{m}$, the length of each sloping edge is $3.4\,\mathrm{m}$, and the distance between each vertex on the base and M is $2\,\mathrm{m}$, as shown in the diagrams.

diagrams not to scale

3D view of tent



2D view of base

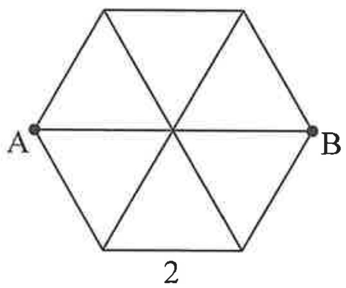


The top of the tent, T, will be supported by a vertical pole from M.

(a) Find the length of the pole, MT. [2]

The hexagonal base can be divided into six equilateral triangles.

diagram not to scale



(b) Find

(i) the area of the base

(ii) the volume of the tent. [5]

(c) Find the value of $\mathrm{M\hat{A}T}$. [2]

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(Question 4 continued)

For extra support, Gaurika decides to attach a rope, with length $2.6\,\text{m}$, to the tent at a point, X, on the edge AT.

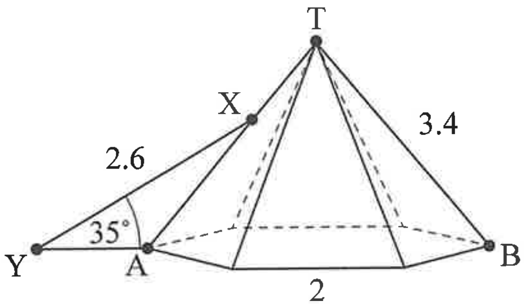
The rope will be fixed to the ground at point Y, such that:

- the rope, [XY], is straight
- points Y, A and B lie on a straight line
- $\text{A}\hat{\text{Y}}\text{X} = 35°$.

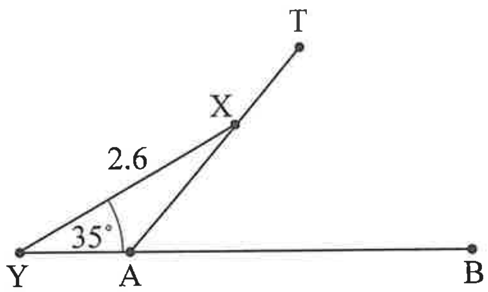
This is shown in the diagrams.

diagrams not to scale

3D view of tent and rope



2D view of triangle AXY



(d) Find AY. [4]

For decoration at night, Gaurika wants to fix a strip of lights from point A to a point, Z, along the rope [XY].

The strip of lights, [AZ], is straight and has length $0.9\,\text{m}$.

(e) Find the two possible values of YZ. [4]

5. [Maximum mark: 15]

In this question, all distances are in kilometres and t is in hours.

Let $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ be a displacement of $1\,\text{km}$ due east, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ be a displacement of $1\,\text{km}$ due north,

and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ be a vertical displacement of $1\,\text{km}$.

Highway 85 in Saudi Arabia is a long, straight, flat road.

Relative to the centre of the town Arar, point O, the position vector of a car, C, travelling along this road is given by:

$$\overrightarrow{\text{OC}} = \begin{pmatrix} 10 \\ -5 \\ 0 \end{pmatrix} + t\begin{pmatrix} 50 \\ -33 \\ 0 \end{pmatrix}.$$

(a) Find the speed of the car. [2]

The police are testing a long-range drone, D, to monitor cars travelling along this road. The drone is launched at $t = 0$ from the point with position vector $\begin{pmatrix} 200 \\ -100 \\ 0.02 \end{pmatrix}$ and flies in a straight line with a constant height of $0.02\,\text{km}$ and a constant velocity of $\begin{pmatrix} -15 \\ -20 \\ 0 \end{pmatrix}$.

(b) Find the angle between the path of the car and the path of the drone. [3]

(c) Write down the position vector, $\overrightarrow{\text{OD}}$, of the drone at time t. [1]

(This question continues on the following page)

(Question 5 continued)

(d) At time t_1, the drone passes through the point with position vector $\begin{pmatrix} 152 \\ p \\ 0.02 \end{pmatrix}$.

Find the value of

(i) t_1

(ii) p. [3]

(e) (i) Find an expression for $\overrightarrow{CD}$, the relative position of the drone from the car.

(ii) Hence, find the shortest distance between the car and the drone. [6]

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6. [Maximum mark: 13]

Juan is creating animations for a website. He uses matrices to transform objects relative to the origin, O.

One matrix that he uses is $A = \begin{pmatrix} \cos(15°) & -\sin(15°) \\ \sin(15°) & \cos(15°) \end{pmatrix}$.

(a) Describe fully the transformation represented by matrix A. [1]

(b) Find the smallest value of n such that $A^n = I$. [2]

Juan also uses matrix B, which represents an enlargement with a scale factor of 1.05, centre $(0, 0)$.

(c) (i) Write down matrix B.

(ii) Describe fully the transformation represented by B^n, where n is the value found in part (b). [3]

Juan creates a new matrix, $C = AB$.

(d) Find matrix C. [2]

Juan creates an animation by repeatedly transforming an object by C.

A point, P, on the object is initially at $(1, 0)$. Juan sets the speed of the animation to 6 transformations per second.

(e) Sketch the path of P for the first 4 seconds of motion **and** label the coordinates of the start and end points. [2]

Juan uses a different transformation, T, defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

To create his animation, he repeatedly transforms an object by T. After many transformations, he notices that all points, (x, y), on the object tend towards a single point, (p, q), such that

$$\lim_{a \to \infty} T^a \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}, \text{ where } a \in \mathbb{Z}^+.$$

(f) Find $\begin{pmatrix} p \\ q \end{pmatrix}$, where $p, q \in \mathbb{R}$. [3]

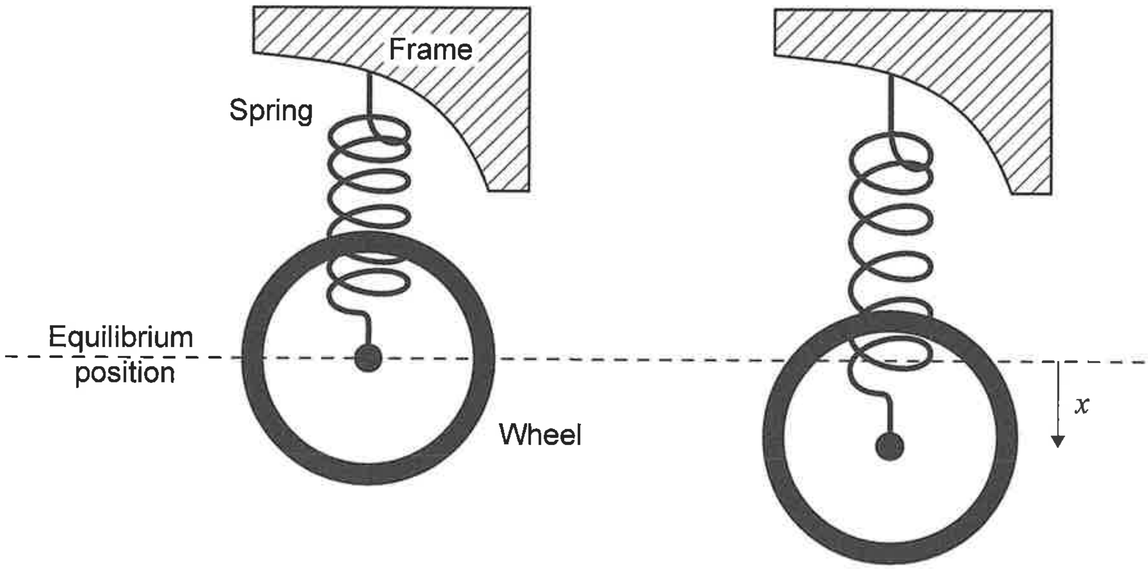
7. [Maximum mark: 18]

The wheel on a motorbike is attached to the frame by a spring. The movement of the spring acts as a **shock absorber**. When the rider sits on the motorbike, the spring compresses, and this position is called the equilibrium position.

When the wheel goes into holes or over bumps in the road, the spring will extend or compress to ensure a smooth ride.

Let x denote the vertical displacement, in centimetres, of the centre of the wheel below the equilibrium position, as shown in the diagram.

diagram not to scale



The vertical displacement of the centre of the wheel, at time t seconds, can be modelled by the differential equation

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0\text{, where } a, b \in \mathbb{R}.$$

Let $y = \frac{dx}{dt}$.

(a) Show that $\frac{dy}{dt} = -bx - ay$. [2]

(This question continues on the following page)

(Question 7 continued)

The equations $y = \frac{\mathrm{d}x}{\mathrm{d}t}$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = -bx - ay$ can be written in matrix form as

$$\begin{pmatrix} \frac{\mathrm{d}x}{\mathrm{d}t} \\ \frac{\mathrm{d}y}{\mathrm{d}t} \end{pmatrix} = \boldsymbol{M}\begin{pmatrix} x \\ y \end{pmatrix}, \text{ where } \boldsymbol{M} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}.$$

A manufacturer wants to compare two springs, Spring 1 and Spring 2, that could be used as shock absorbers.

The differential equation for Spring 1 has $a = 18$ and $b = 77$.

For these values, the eigenvalues of $\boldsymbol{M}$ are -7 and -11.

(b) Find the corresponding eigenvectors. [3]

The manufacturer models **both** springs using the same initial conditions:

$$t = 0,\ x = 5\,\mathrm{cm} \text{ and } \frac{\mathrm{d}x}{\mathrm{d}t} = 2\,\mathrm{cm\,s^{-1}}.$$

(c) Hence, for Spring 1

(i) find the exact solution for $x(t)$

(ii) sketch the graph of $x(t)$, for $0 \le t \le 1$. [7]

The differential equation for Spring 2 has $a = 18$ and $b = 85$.

For these values, the eigenvalues of $\boldsymbol{M}$ are $-9 \pm 2\mathrm{i}$.

(d) (i) Sketch the phase portrait for Spring 2, indicating the direction of the trajectory.

(ii) Hence, sketch the graph of x against t. [5]

(e) Using your answers to parts (c)(ii) and (d)(ii), give a reason why Spring 1 might make a better shock absorber than Spring 2. [1]

Disclaimer:

References:

1. Morton, C., 2022. *The Best Countries for Expats—According to Expats*. [online] Available at: https://www.cntraveler.com/gallery/the-10-best-countries-for-expats [Accessed 13 November 2023]. SOURCE ADAPTED.

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